Different aspects of the Tower of Hanoi game

Sandi Klavžar

Fakulteta za matematiko in fiziko, Univerza v Ljubljani

Fakulteta za naravoslovje in matematiko, Univerza v Mariboru

Dipartimento di Mathematica e Informatica
Università degli Studi di Trieste
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Classical problem

Édouard Lucas: the cover plate of the Tower of Hanoi from 1883

N. Claus (de Siam)

Lucas d'Amiens

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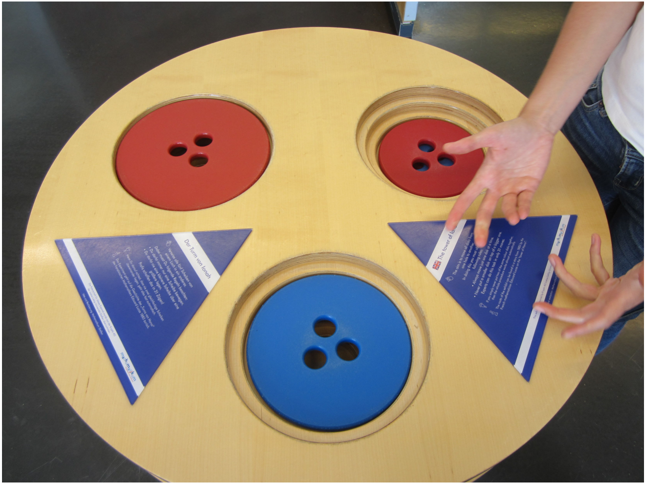
- Move/second: $5.849424 \cdot 10^{11}$ years = 585 billion years.

Mathematikum Gießen - exponential growth

Mathematikum Gießen - “The Tower of Ionah”



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Procedure ToH(n, i, j)
Parameter n: number of discs
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- An edge represents a move of a discs from one peg to another:

$$E(H_3^n) = \left\{\{\underline{s}i(3-i-j)^{d-1}, \underline{s}j(3-i-j)^{d-1}\} \mid \underline{s} \in \{0,1,2\}^{n-d}\right\} .$$

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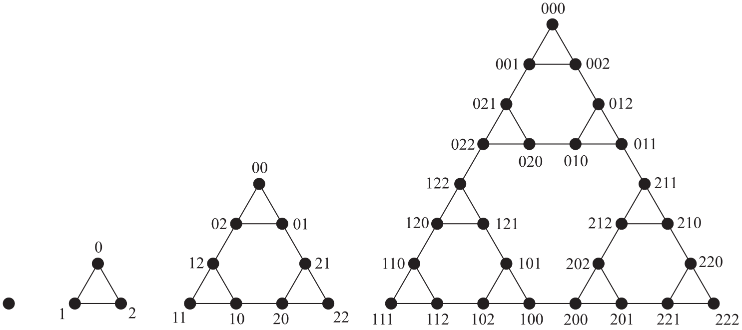
- $|E(H_3^n)| = \frac{3}{2}(3^n - 1)$.

Hanoi graphs -cont'd



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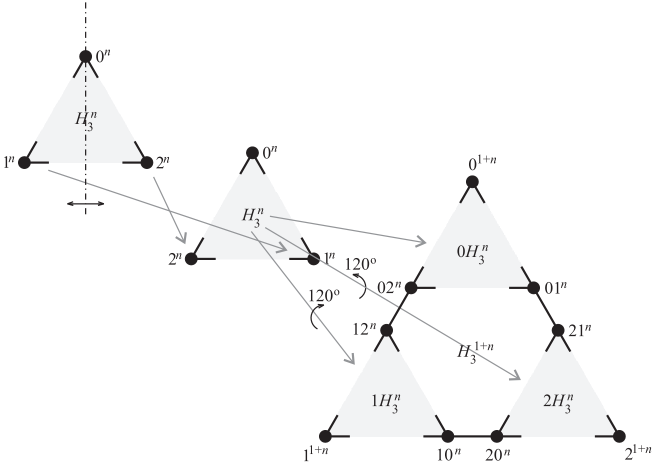

Variations of the Puzzle

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- Discs are distinguishable.
- Discs are on pegs all the time except for moves.
- One or more discs can only be moved from the top of a stack.
- Task: given an initial distribution of discs among pegs and a goal distribution of discs among pegs, find a shortest sequence of moves that transfers discs from the initial state to the final state obeying the rules.

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- More than one top disc may be moved in a single move.
- There can be additional restrictions or relaxations on moves, the latter even violating the divine rule.
- And, of course, any combination of the above.

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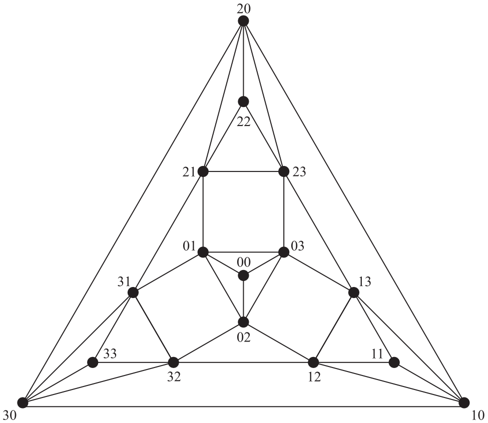
- Thierry Bousch, 2014, solved the problem for four pegs!
- Which task is most demanding? Korf phenomenon: $n = 15, 20$!

The Hanoi graph H_4^2



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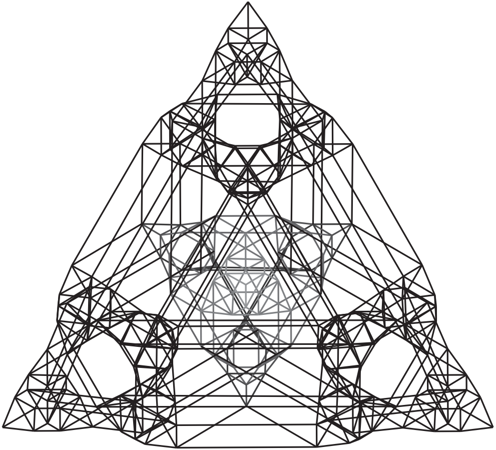
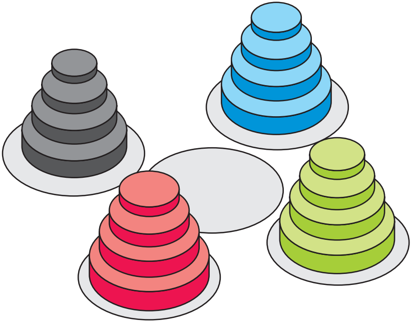


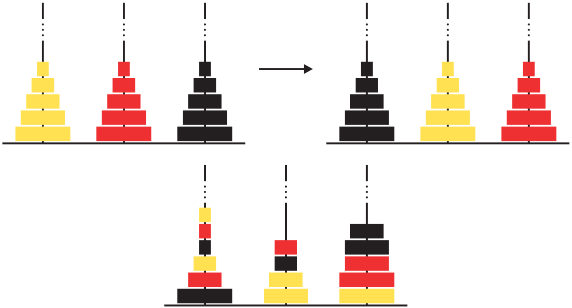
Lucas variant from 1889

- 16 discs of mutually different sizes.
- Task: transfer all discs onto the middle peg: $(1234)^4 \to 0^{16}$.
- Optimal solution has 63 moves (computer experiment).
- Second task: $1^4 2^4 3^4 4^4 \to 0^{16}$.
- Optimal solution has 54 moves (computer experiment).



The Tower of Antwerpen

- 3 pegs, $3 \times n$ identical discs—except in color.
- A peg can hold an arbitrary number of discs.
- Discs of the same size may be put on top of each other.
- Task: each tower rests on a different peg than originally.
- Theorem: the TA puzzle with $3n$ discs can be solved in the optimal number of $3 \cdot 2^{n+2} - 8n - 10$ moves.



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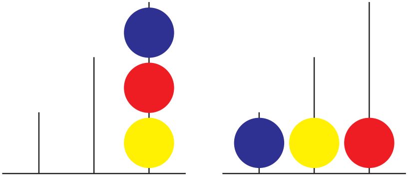
- Goal: reach a specified state from another designated state in the minimum number of moves.



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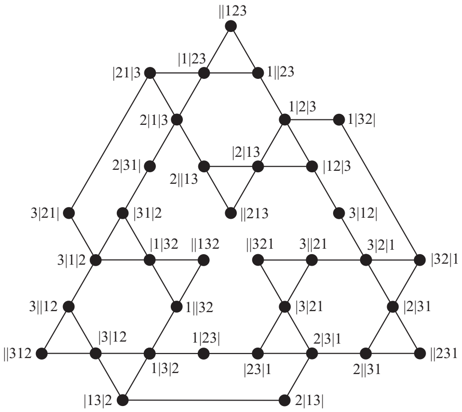
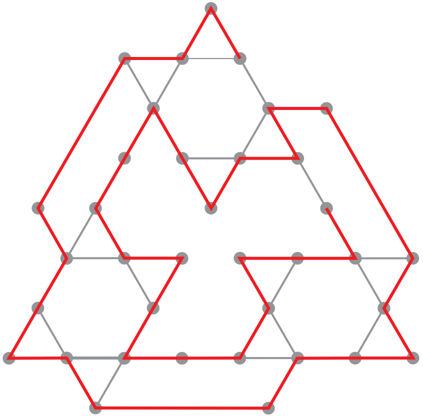
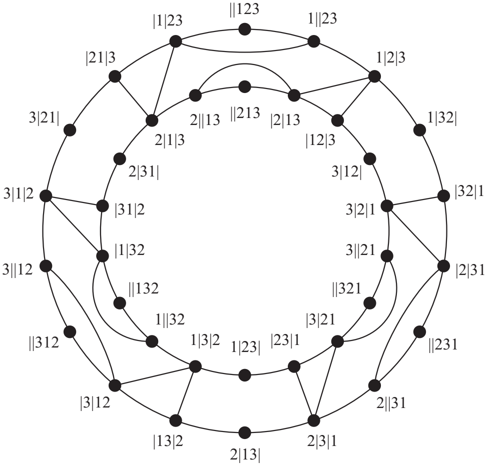


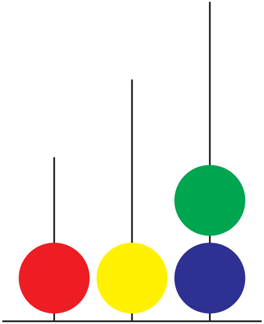
Hamiltonian path in L



The graph L again



London tower - generalization L_h^n (L_{234}^4)



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- Short description: $TH(D)$.

Oriented disc moves - cont'd

$TH(D)$ is solvable if for any choice of source and goal pegs and for every number of discs there exists a sequence of legal moves.

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The Linear TH is solvable. Its state graph is the path on 3^n vertices between the perfect states on pegs 0 and 2. In particular, the optimal solution for any task is unique.

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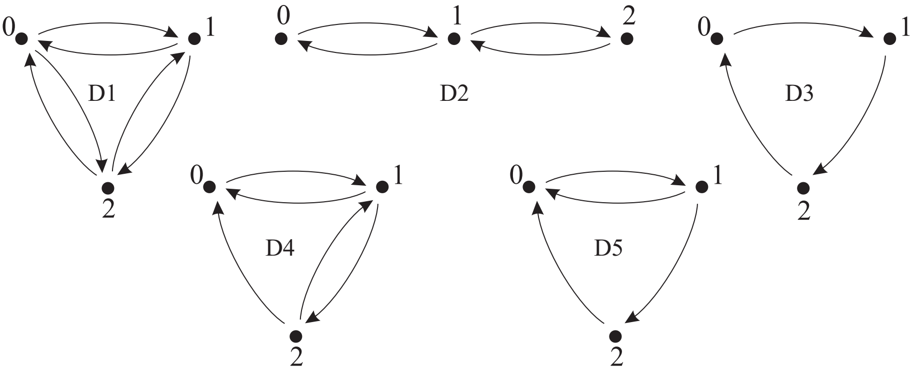
Theorem

Let $D = (V(D), A(D))$ be a digraph with at least three vertices. Then $TH(D)$ is solvable if and only if D is strong.

Strongly connected digraphs of order 3



More on the classical task

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- How to reach a perfect state from an irregular state?

Non-repetitive sequences

A sequence $a = (a_n)_{n\in\mathbb{N}}$ of symbols a_n from an alphabet A is called non-repetitive or square-free (over A) if it does not contain a subsequence

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$$a_{i+j} = a_{i+j+m}, \quad j = 1, \dots, m\,.$$

Examples

- $1, 2, 1, 3, 1, 2, 3, 1, 2, 1$

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Examples

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- $1, 2, 1, \boxed{3,\ 1,\ 2,}\boxed{3,\ 1,\ 2,}1$
- 1, 2, 3, 4, 5, 6, ...
- Let's find a non-repetitive sequence with the greedy strategy:
- 1, 2, 1, 3, 1, 2, 1, 4,

Examples

- $1, 2, 1, 3, 1, 2, 3, 1, 2, 1$
- $1, 2, 1, \boxed{3,\ 1,\ 2,}\boxed{3,\ 1,\ 2,}1$
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Examples

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- $1, 2, 1, \boxed{3,\ 1,\ 2,}\boxed{3,\ 1,\ 2,}1$
- 1, 2, 3, 4, 5, 6, . . .
- Let's find a non-repetitive sequence with the greedy strategy:
- 1, 2, 1, 3, 1, 2, 1, 4, 1, 2,

Examples

- $1, 2, 1, 3, 1, 2, 3, 1, 2, 1$
- $1, 2, 1, \boxed{3,\ 1,\ 2,}\boxed{3,\ 1,\ 2,}1$
- $1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots$
- Let's find a non-repetitive sequence with the greedy strategy:
- $1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1,\ 4,\ 1,\ 2,\ 1,$

Examples

- $1, 2, 1, 3, 1, 2, 3, 1, 2, 1$
- $1, 2, 1, \boxed{3,\ 1,\ 2,}\boxed{3,\ 1,\ 2,}1$
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- Let's find a non-repetitive sequence with the greedy strategy:
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Examples

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- $1, 2, 1, \boxed{3,\ 1,\ 2,}\boxed{3,\ 1,\ 2,}1$
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- Let's find a non-repetitive sequence with the greedy strategy:
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Examples

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- $1, 2, 1, \boxed{3,\ 1,\ 2,}\boxed{3,\ 1,\ 2,}1$
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Examples

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- $1, 2, 1, \boxed{3,\ 1,\ 2,}\boxed{3,\ 1,\ 2,}1$
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- Let's find a non-repetitive sequence with the greedy strategy:
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Examples

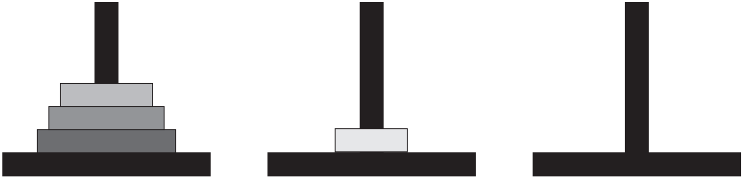
- $1, 2, 1, 3, 1, 2, 3, 1, 2, 1$
- $1, 2, 1,$ $\boxed{3,\ 1,\ 2,}$ $\boxed{3,\ 1,\ 2,}$ 1
- 1, 2, 3, 4, 5, 6, . . .
- Let's find a non-repetitive sequence with the greedy strategy:
- 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5,

Examples

- $1, 2, 1, 3, 1, 2, 3, 1, 2, 1$
- $1, 2, 1,$ $\boxed{3,\ 1,\ 2,}\boxed{3,\ 1,\ 2,}$ 1
- 1, 2, 3, 4, 5, 6, . . .
- Let's find a non-repetitive sequence with the greedy strategy:
- 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, . . .

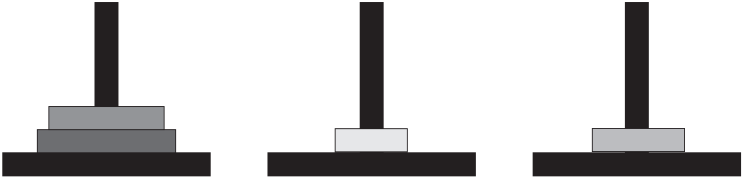
The classical task with 4 discs

The classical task with 4 discs



1

The classical task with 4 discs

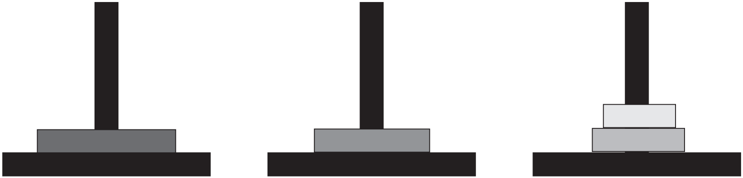


1 2

The classical task with 4 discs

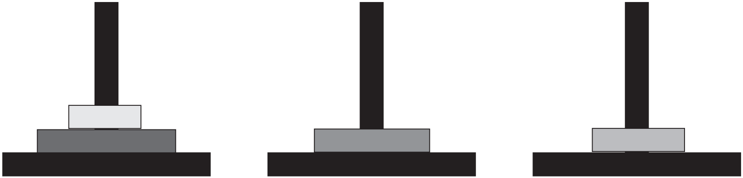
1 2 1

The classical task with 4 discs



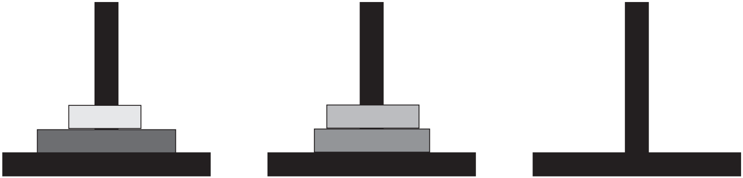
1 2 1 3

The classical task with 4 discs



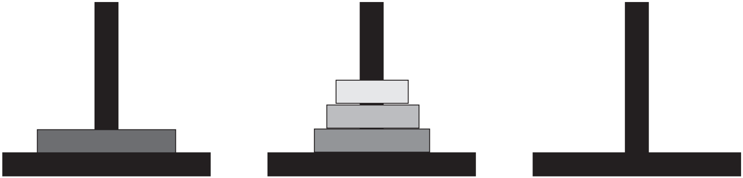
1 2 1 3 1

The classical task with 4 discs



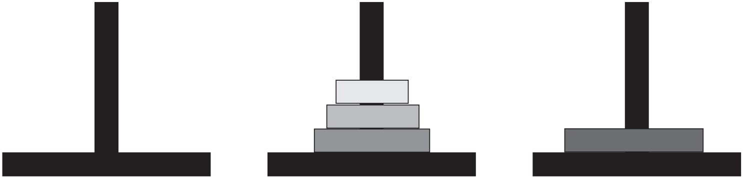
1 2 1 3 1 2

The classical task with 4 discs



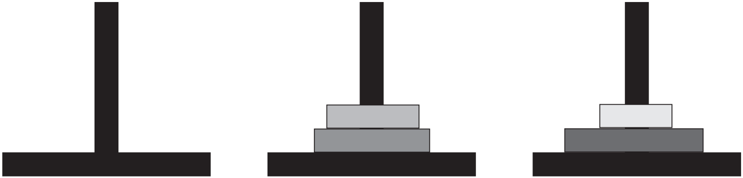
1 2 1 3 1 2 1

The classical task with 4 discs



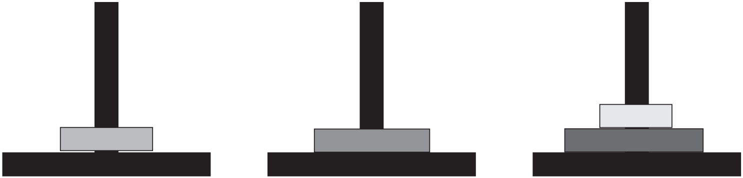
1 2 1 3 1 2 1 4

The classical task with 4 discs



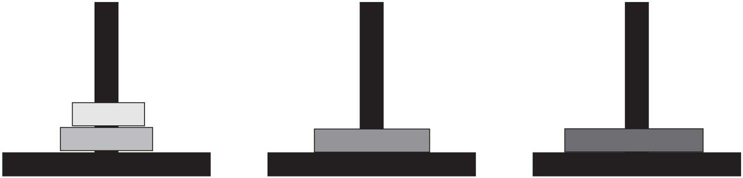
1 2 1 3 1 2 1 4 1

The classical task with 4 discs



1 2 1 3 1 2 1 4 1 2

The classical task with 4 discs

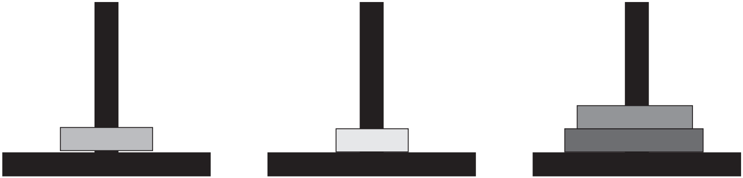


1 2 1 3 1 2 1 4 1 2 1

The classical task with 4 discs

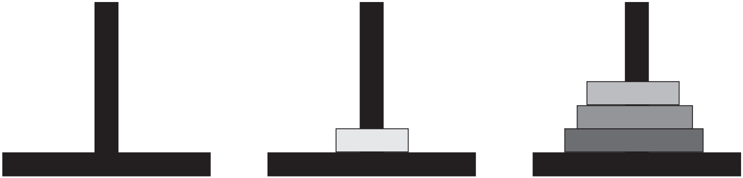
1 2 1 3 1 2 1 4 1 2 1 3

The classical task with 4 discs



1 2 1 3 1 2 1 4 1 2 1 3 1

The classical task with 4 discs



1 2 1 3 1 2 1 4 1 2 1 3 1 2

The classical task with 4 discs

1 2 1 3 1 2 1 4 1 2 1 3 1 2 1

The classical task with 4 discs cont'd

Let's code moves as follows:

The classical task with 4 discs cont'd

Let's code moves as follows:

- Move from peg 1 to 2: a

The classical task with 4 discs cont'd

Let's code moves as follows:

- Move from peg 1 to 2: a
- Move from peg 2 to 3: b

The classical task with 4 discs cont'd

Let's code moves as follows:

- Move from peg 1 to 2: a
- Move from peg 2 to 3: b
- Move from peg 3 to 1: c

The classical task with 4 discs cont'd

Let's code moves as follows:

- Move from peg 1 to 2: a
- Move from peg 2 to 3: b
- Move from peg 3 to 1: c
- Move from peg 2 to 1: $\bar{a}$

The classical task with 4 discs cont'd

Let's code moves as follows:

- Move from peg 1 to 2: a
- Move from peg 2 to 3: b
- Move from peg 3 to 1: c
- Move from peg 2 to 1: $\bar{a}$
- Move from peg 3 to 2: $\bar{b}$

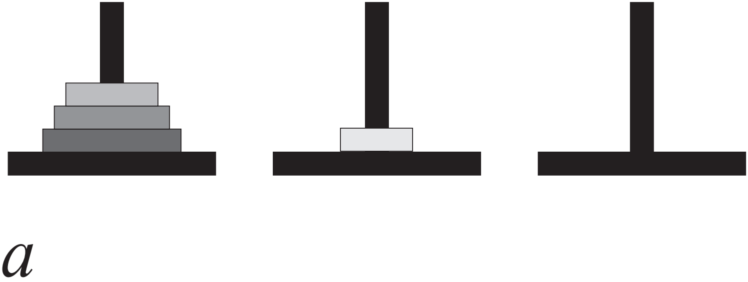
The classical task with 4 discs cont'd

Let's code moves as follows:

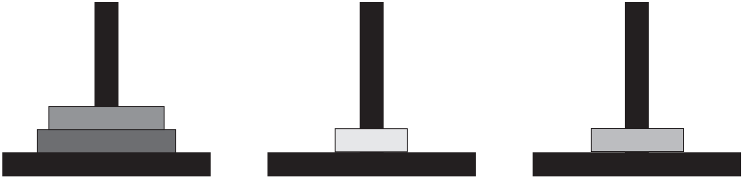
- Move from peg 1 to 2: a
- Move from peg 2 to 3: b
- Move from peg 3 to 1: c
- Move from peg 2 to 1: $\bar{a}$
- Move from peg 3 to 2: $\bar{b}$
- Move from peg 1 to 3: $\bar{c}$

The classical task with 4 discs cont'd

The classical task with 4 discs cont'd



The classical task with 4 discs cont'd

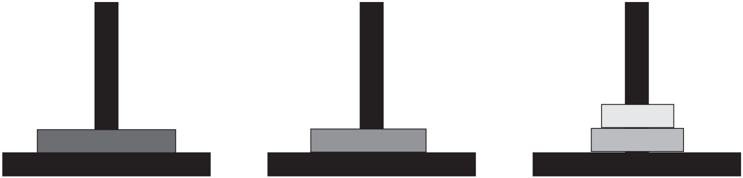


$a\ \bar{c}$

The classical task with 4 discs cont'd

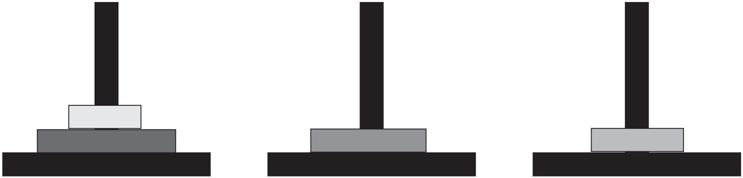
$a\ \bar{c}\ b$

The classical task with 4 discs cont'd



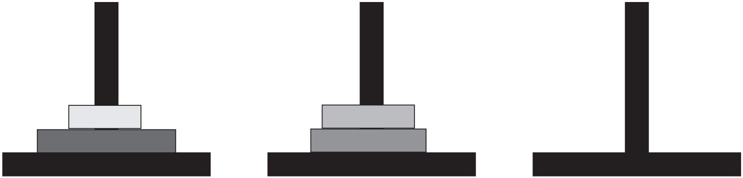
$a\ \bar{c}\ b\ a$

The classical task with 4 discs cont'd



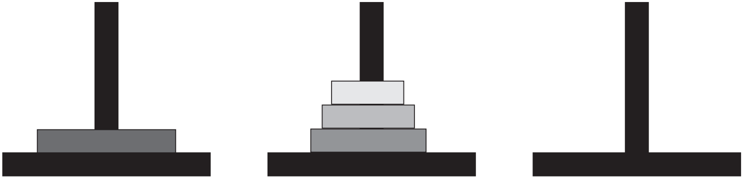
$a\ \bar{c}\ b\ a\ c$

The classical task with 4 discs cont'd



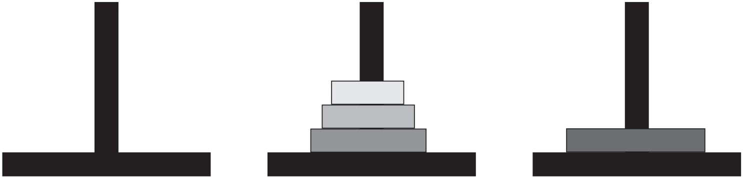
$a\ \bar{c}\ b\ a\ c\ \bar{b}$

The classical task with 4 discs cont'd



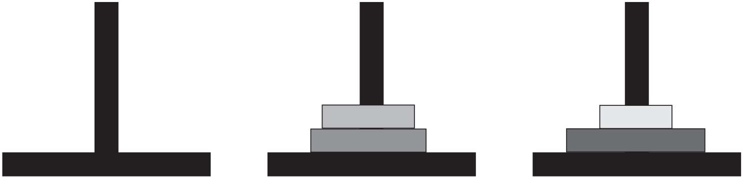
$a\ \bar{c}\ b\ a\ c\ \bar{b}\ a$

The classical task with 4 discs cont'd



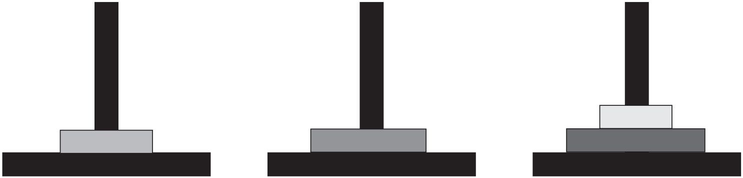
$$a\ \bar{c}\ b\ a\ c\ \bar{b}\ a\ \bar{c}$$

The classical task with 4 discs cont'd



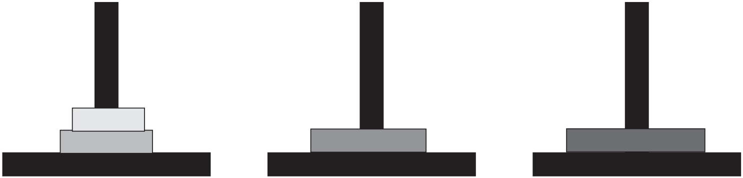
$a\ \bar{c}\ b\ a\ c\ \bar{b}\ a\ \bar{c}\ b$

The classical task with 4 discs cont'd



$a\ \bar{c}\ b\ a\ c\ \bar{b}\ a\ \bar{c}\ b\ \bar{a}$

The classical task with 4 discs cont'd

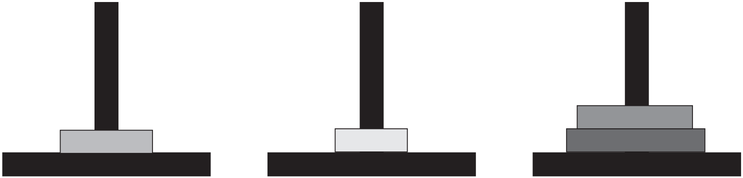


$$a\ \bar{c}\ b\ a\ c\ \bar{b}\ a\ \bar{c}\ b\ \bar{a}\ c$$

The classical task with 4 discs cont'd

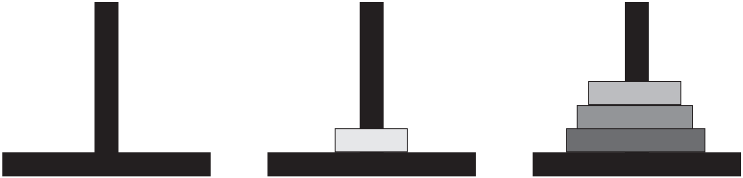
$$a\ \bar{c}\ b\ a\ c\ \bar{b}\ a\ \bar{c}\ b\ \bar{a}\ c\ b$$

The classical task with 4 discs cont'd



$$a\ \bar{c}\ b\ a\ c\ \bar{b}\ a\ \bar{c}\ b\ \bar{a}\ c\ b\ a$$

The classical task with 4 discs cont'd



$$a\ \bar{c}\ b\ a\ c\ \bar{b}\ a\ \bar{c}\ b\ \bar{a}\ c\ b\ a\ \bar{c}$$

The classical task with 4 discs cont'd

$$a\ \bar{c}\ b\ a\ c\ \bar{b}\ a\ \bar{c}\ b\ \bar{a}\ c\ b\ a\ \bar{c}\ b$$

6 symbols suffice

6 symbols suffice

Theorem (Allouche, Astoorian, Randall, Shallit, 1994)

ToH sequence

$$a, \bar{c}, b, a, c, \bar{b}, a, \bar{c}, b, \bar{a}, c, b, a, \bar{c}, b, \ldots$$

is non-repetitive.

There is more

Consider

$$(a, \bar{c}, b), (a, c, \bar{b}), (a, \bar{c}, b), (\bar{a}, c, b), (a, \bar{c}, b), \ldots$$

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Consider

$$(a, \bar{c}, b), (a, c, \bar{b}), (a, \bar{c}, b), (\bar{a}, c, b), (a, \bar{c}, b), \ldots$$

There exists exactly five types of such triples:

$$(a, \bar{c}, b) \quad (a, c, \bar{b}) \quad (\bar{a}, c, b) \quad (a, c, b) \quad (\bar{a}, c, \bar{b})\,.$$

There is more

Consider

$$(a, \bar{c}, b), (a, c, \bar{b}), (a, \bar{c}, b), (\bar{a}, c, b), (a, \bar{c}, b), \ldots$$

There exists exactly five types of such triples:

$$(a, \bar{c}, b) \quad (a, c, \bar{b}) \quad (\bar{a}, c, b) \quad (a, c, b) \quad (\bar{a}, c, \bar{b})\,.$$

Therefore:

Theorem (Hinz, 1996)

ToH yields an infinite non-repetitive sequence using five symbols only.

Thank you for your attention!